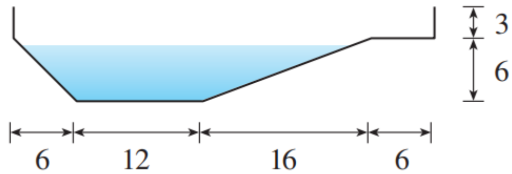
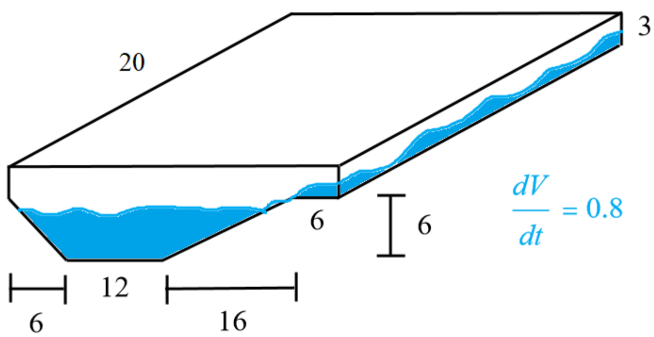
Exercise 28

A swimming pool is 20 ft wide, 40 ft long, 3 ft deep at the shallow end, and 9 ft deep at its deepest point. A cross-section is shown in the figure. If the pool is being filled at a rate of 0.8 ft^3/min, how fast is the water level rising when the depth at the deepest point is 5 ft?

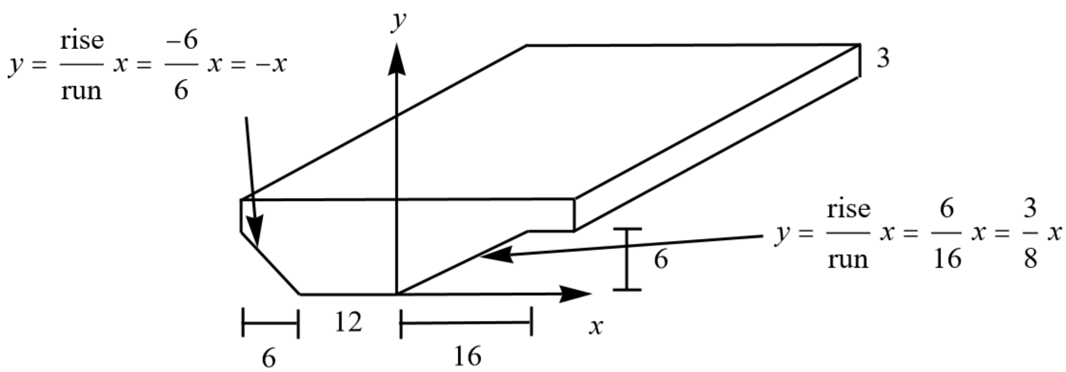


Solution

Start by drawing a schematic of the trough at a certain time.

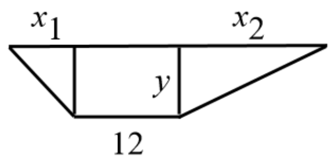


Write equations for the lines representing the trapezoid's edges.



The aim is to find the volume $V(y)$ that water occupies if it's at a height y that's less than 6 feet.

In order to do this, find the area of a cross-section and then multiply it by 20 feet, the thickness.



This area consists of two triangles and a rectangle.

$$A = \frac{1}{2}x_1 y + 12y + \frac{1}{2}x_2 y$$

Since we want to find dy/dt when $y = 5$, eliminate x_1 and x_2 in favor of y.

$$A = \frac{1}{2}(y)y + 12y + \frac{1}{2}\left(\frac{8y}{3}\right)y$$

$$= \frac{11y^2}{6} + 12y$$

Multiply the area by the thickness, 20 feet, to get the volume.

$$V = 20A$$

$$= 20\left(\frac{11y^2}{6} + 12y\right)$$

$$= \frac{110y^2}{3} + 240y$$

Take the derivative of both sides with respect to t by using the chain rule.

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{110y^2}{3} + 240y\right)$$

$$\frac{dV}{dt} = \frac{220y}{3} \cdot \frac{dy}{dt} + 240 \cdot \frac{dy}{dt}$$

$$0.8 = \left(\frac{220y}{3} + 240\right)\frac{dy}{dt}$$

Solve for dy/dt.

$$\frac{dy}{dt} = \frac{0.8}{\frac{220y}{3} + 240}$$

Therefore, the rate that the water level is rising when the water is 5 feet deep is

$$\left.\frac{dy}{dt}\right|_{y=5} = \frac{0.8}{\frac{220(5)}{3} + 240} = \frac{3}{2275} \approx 0.00131868 \, \frac{\text{ft}}{\text{min}}.$$